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WEAK SOLUTION CLASSES FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

We study partial integro-differential equations of the type

$$\text{(I)}\qquad \partial_t u(\cdot,t) + Au(\cdot,t) + \int_0^t a(t-s)Bu(\cdot,s)ds = f(\cdot,t),\ 0 < t < T,$$

in some spatial domain $\Omega \subset \mathbb{R}^n$, A being a linear and B a quasilinear elliptic operator of second order, both in divergence form, together with initial and various boundary conditions. We give conditions on the structure of A and B that lead to a priori estimates and show how to get the existence of weak solutions $(u(\cdot,t) \in W^{1,p}(\Omega)$ or $u(\cdot,t) \in W^{2,2}_{loc}(\Omega)$ for a.e. $t)$ from approximating solutions (that solve finite-dimensional versions of (I) or versions with modified coefficients). The main tools are "energy" estimates on $\|\partial_t u(\cdot,t)\|^2_{L^2} + \int_\Omega G(\nabla_x u)$, if $Bu = -\mathrm{div}_x(\nabla_u G(\nabla_x u))$, for $W^{1,p}$-solutions, and estimates on the L^2-product $\langle Au,Bu\rangle_{L^2}$ for $W^{2,2}_{loc}$-solutions.

AMS (MOS) Subject Classifications: 35K60, 45K05, 73F15

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SIGNIFICANCE AND EXPLANATION

This paper studies a class of integro-differential equations that arises in some models for heat conduction in materials with memory or for the deformation of visco-elastic membranes. Some classes of constitutive assumptions are given that ensure the existence of weak solutions for these models; i.e., stress or heat flux are integrable fields over the reference configuration. The models are hybrids between damped nonlinear wave equations and perturbed heat equations, and mathematical techniques for these different problems are combined to establish existence results.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

WEAK SOLUTION CLASSES FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

Hans Engler* and Stephan Luckhaus*

1. Introduction

In this paper we want to consider the integro-differential equation

$$\text{(I)} \qquad \partial_t u(x,t) - \Delta_x u(x,t) - \int_0^t a(t-s)\operatorname{div}_x g(\nabla_x u(x,s))ds = f(x,t)$$

$$\text{in} \quad \Omega \times (0,T)$$

together with an initial condition

$$\text{(1.0)} \qquad u(\cdot,0) = u^0 \quad \text{in} \quad \Omega$$

and boundary conditions

$$\text{(1.1)} \qquad u \equiv u^1 \quad \text{on} \quad (\partial\Omega\backslash\Gamma) \times [0,T] \ ,$$

$$\text{(1.2)} \qquad -\nu \cdot (\nabla_x u(x,t) + \int_0^t a(t-s)g(\nabla u(x,s))ds) = \beta(u(x,t))$$

$$\text{on} \quad \Gamma \times (0,T).$$

Here $\Omega \subset \mathbb{R}^n$ is bounded with Lipschitz boundary $\partial\Omega$, $\Gamma \subset \partial\Omega$, ν is the outward normal. The function $g : \mathbb{R}^n \to \mathbb{R}^n$ is a gradient, subject to certain growth conditions; a is a scalar kernel with some regularity properties and $a(0) = 1$; β is a monotone function. The functions u^0, u^1 are traces of some function $u_0 : \bar{\Omega} \times [0,T] \to \mathbb{R}$, f and u_0 are in certain regularity classes. The precise assumptions are stated in the sections below.

In Section 2 we prove the existence of distributional solutions, using a version of a technique that has been used by J. Clements ([4]) for the case $a \equiv 1$ and constant Dirichlet boundary conditions. In Section 3 we consider specifically the "isotropic" case $g(\xi) = g_0(|\xi|) \cdot \xi$ and prove some results on inner regularity, showing that all terms

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appearing in (I) are in $L^1_{loc}(\Omega \times [0,T])$. For the case of constant Dirichlet data and a domain Ω with C^2-boundary, it is shown in Section 4 that the regularity estimates hold up to the boundary. Sections 3 and 4 use a device by V. Barbu ([2]) and M. Crandall/S.-O. Londen/J. Nohel ([6]). No claims concerning the uniqueness of the solution are made in the general case; for this question and some other remarks see Section 4.

Equation (I) has a physical interpretation from the theory of heat conduction in materials with memory. Consider a homogeneous rigid heat-conducting material occupying some region $\Omega \subset R^3$. Let q denote the heat flux, u the absolute temperature and e the internal energy. In various general models for heat conduction (cf. [5], [17], [19]) it has been proposed that q and e should depend both on the present value and the history of the temperature and its gradient. The constitutive assumptions

$$q(x,t) = -a_0 \nabla_x u(x,t) - \int_0^\infty a(s) g(\nabla_x u(x,t-s))ds , \tag{1.3}$$

$$e(x,t) = e_0(x) + \kappa \cdot u(x,t) \tag{1.4}$$

($\kappa > 0$ and $a_0 > 0$ denoting heat capacity resp. conductivity, a a suitable relaxation kernel) together with the law of energy balance

$$\partial_t e(x,t) + \mathrm{div}_x q(x,t) = r(x,t) \tag{1.5}$$

(r denoting heat sources or sinks) then give (I) after rescaling time and prescribing the temperature history u up to $t = 0$. The boundary condition (1.1) corresponds to a fixed temperature outside of Ω and perfect heat conduction through the boundary; (1.2) corresponds, e.g., to a radiation law or to local temperature control at the boundary (cf. [9]). This physical model leads us to regard (I) as a perturbed heat equation.

Another physical interpretation of (I) comes from the theory of viscoelastic materials: The one-dimensional version of (I) with $a \equiv 1$ describes longitudinal motions of a homogeneous bar composed of a Kelvin solid (cf. [21], [10]), assuming the following relation between strain E and Picla-Kirchhoff stress Σ:

$$\Sigma = G(E) + L(\dot{E}) , \tag{1.6}$$

L a linear tensor-valued function, $\dot{E}$ denoting the time derivative of E. The twodimensional equation (I) then arises in a model for the normal displacement u of a membrane composed of such a material. The boundary condition (1.1) corresponds to a fixed portion of the edge of the membrane, (1.2) can be interpreted as a friction-type boundary condition, the friction coefficient depending on the displacement. It should be noted, however, that one would have to take E to be the linear infinitesimal strain in order to arrive at (I), which somewhat disagrees with taking G as a general non-linear function in (1.6). Nevertheless this leads us to view (I) as a damped non-linear wave equation.

It should be noted that the fundamental differences between these two physical interpretations essentially appear in the asymptotic properties of the kernel a and the forcing term f; cf. [18] for a discussion of these problems.

Various authors have discussed the one-dimensional version of the visco-elastic model problem leading to (I) (hence $a \equiv 1$) and shown existence, uniqueness, and asymptotic properties of classical solutions ([1], [7], [12], [23]). Weak solutions of the more general equation (I) (a arbitrary, $n = 1$) have been discussed in [20] and as applications of abstract theorems in [2] and [6]. The n-dimensional case for $a \equiv 1$ and homogeneous Dirichlet boundary data has been treated in [4] where distributional solutions are shown to exist.

A few words on the notation that we are going to employ:

For $x \in \mathbb{R}^k$, $|x|$ denote the norm; $\|\cdot\|$ is reserved to Banach space norms.

For $f : \mathbb{R}^n \to \mathbb{R}^\ell$, $\nabla f = \nabla_x f$ is the matrix of (weak) derivatives wherever it exists (gradient for $\ell = 1$); $\mathrm{div}_x f = \mathrm{div}\, f$ is the divergence operator applied to f, if $n = \ell$. For $\Omega \subset \mathbb{R}^n$, $W^{k,p}(\Omega,X)$ is the usual Sobolev space (for $X = \mathbb{R}$ or $X = \mathbb{R}^\ell$ or X a Banach space); $C_0^k(\Omega)$ is the space of C^k-functions $f : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(f)$ (the closure taken in Ω) is compact, also if Ω is not open; $W_0^{k,p}(\Omega)$ is the closure of $C_0^k(\Omega)$ with respect to the $W^{k,p}$-norm. Dependence on the variables $x \in \mathbb{R}^n$ or $t \in \mathbb{R}$ is suppressed where no confusion will arise.

By $a * b(t)$, $a \in L^1(0,T;\mathbb{R})$, $b \in L^1(0,T;X)$, X a Banach space, we denote convolution with respect to t:

$$a * b(t) = \int_0^t a(t - s)b(s)ds \ .$$

The symbol C, when appearing in proofs, denotes a constant whose value can change from line to line but which depends only on given properties.

2. Weak Solutions of the Dirichlet Problem.

In this section we want to show the existence of solutions of (I) if Γ is the empty set; i.e. the boundary condition (1.1) should hold on all of $\partial\Omega \times [0,T]$. We shall use the following assumptions:

(A1) The region $\Omega \subset \mathbb{R}^n$ is open and bounded.

(A2) The function $g : \mathbb{R}^n \to \mathbb{R}^n$ is given by $g(\xi) = \nabla_\xi G(\xi)$, $G(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}$ being a C^1-function. There exists a constant $L > 0$, such that $\tilde{G}(\xi) = G(\xi) + \frac{L}{2}(|\xi|^2 + 1)$ is convex and positive, and there exists a $C_0 > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$

$$(2.1) \qquad |g(\xi) \cdot \eta| \le C_0 \cdot (\tilde{G}(\xi) + \tilde{G}(\eta) + 1) .$$

(A3) The kernel a is in $W^{2,1}([0,T],\mathbb{R})$; $a(0) = 1$.

(A4) The function $u_0 : \bar{\Omega} \times [0,T] \to \mathbb{R}$ satisfies

$$\int_0^T \int_\Omega \{\tilde{G}(\nabla\partial_s u_0(\cdot,s)) + |\nabla_x \partial_s u_0(\cdot,s)|^2 + |\partial_s^2 u_0(\cdot,s)|^2\} ds < \infty ,$$

$$u_0(\cdot,0) \in W^{2,2}(\Omega) ,$$

and

$$\int_\Omega \tilde{G}(\nabla_x u_0(\cdot,0)) < \infty .$$

(A5) The function f is in $W^{1,1}([0,T], L^2(\Omega))$.

We are going to prove the following result:

<u>Theorem 2.1</u>: Suppose (A1) through (A5) hold. Then the equation (I) together with initial and boundary conditions (1.0), (1.1) has a distributional solution u; i.e. u satisfies

$$(2.2) \qquad \int_0^T \int_\Omega \{(\nabla_x u + a * g(\nabla_x u)) \cdot \nabla_x \phi - u \cdot \partial_t \phi - f \cdot \phi\} dxdt = \int_\Omega u_0(\cdot,0) \cdot \phi(\cdot,0) dx$$

for all test functions $\phi \in C_0^1(\Omega \times [0,T),\mathbb{R})$; and $(u - u_0)(\cdot,t) \in W_0^{1,2}(\Omega,\mathbb{R})$ for a.e. t. Moreover,

$$(2.3)\qquad \int_0^T\int_\Omega |\partial_t\nabla_x u(\cdot,s)|^2 ds + \sup_{[0,T]}\int_\Omega (|\partial_t u(\cdot,t)|^2 + G(\nabla_x u(\cdot,t))) \le K < \infty ,$$

K depending only on the data of the problem.

<u>Proof</u>. We shall use a Galerkin procedure and

1. find approximating solutions,
2. deduce a priori estimates for them,
3. show that some of their weak clusterpoints solve (I).

<u>Step 1</u>: Let $(V_m)_{m\ge 1}$ be a sequence of finite-dimensional subspaces of $W_0^{1,2}(\Omega)$, $\bigcup_m V_m$ be dense in $W_0^{1,2}(\Omega)$, $V_m \subset C^1(\bar\Omega)$. We seek solutions $\tilde u_m : [0,T] \to V_m$ of the systems of ordinary integro-differential equations

$$(2.4)\qquad \int_\Omega \partial_t\tilde u^m(\cdot,t)\cdot v + \int_\Omega (\nabla_x\tilde u^m(\cdot,t) + a * g(\nabla_x\tilde u^m + \nabla_x u_0)(\cdot,t))\cdot\nabla_x v$$

$$= \int_\Omega (f(\cdot,t) - \partial_t u_0(\cdot,t))\cdot v - \int_\Omega \nabla_x u_0(\cdot,t)\cdot\nabla_x v$$

for all $v \in V_m$ and $0 \le t \le T$; $\tilde u^m(\cdot,0) = 0$. By standard theorems on functional differential equations (see [13]), (2.4) has a unique local solution $\tilde u^m : [0,T_m] \to V_m$ for all m; $\tilde u^m$ is of class $C^{1,1}$ with respect to t.

<u>Step 2</u>: Let $u^m = \tilde u^m + u_0$. We show that there exists a constant C^*, depending only on u_0, f, and the properties of a and g, such that for all m

$$(2.5)\qquad \sup_{[0,T_m]}\int_\Omega (G(\nabla_x u^m)(\cdot,t) + |\partial_t u^m(\cdot,t)|^2) + \int_0^T\int_\Omega |\nabla\partial_t u^m(\cdot,t)|^2 dt \le C^* ,$$

which shows also that solutions of (2.4) exist on $[0,T]$.

To show (2.5), we shall transform (2.4) such that $a \equiv 1$, differentiate test with $\partial_t\tilde u^m$, integrate over $[0,t]$, and show that the "good" terms (that appear in (2.5)) dominate all the rest. Let r be the resolvent kernel of $\dot a$; i.e. $r : [0,T] \to \mathbb{R}$ is defined by

$$r(t) + \int_0^t r(t-s)\dot a(s)ds + \dot a(t) = 0, \qquad 0 \le t \le T .$$

Then r is as regular as $\dot{a}$, and for $y,z \in L^1(0,T;\mathbb{R})$

(2.6) $$a * y = z \text{ on } [0,T] \text{ iff } 1 * y = z + r * z \text{ on } [0,T] .$$

We apply (2.6) to (2.4), with

$$y(t) := \int_\Omega g(\nabla_x u^m(\cdot,t)) \cdot \nabla_x v ,$$

differentiate the resulting identity, and note that it is possible to take t-dependent test functions $v \in L^2(0,T;V_m)$. We then choose $v(\tau) = \partial_\tau \tilde{u}^m(\cdot,\tau)$, and integrate from 0 to t. The result can be written in the form

(2.7) $$I_1(t) + I_2(t) + I_3(t) = I_4(t) ,$$

with the following notation:

$$I_1(t) = \frac{1}{2}\int_\Omega |\partial_t \tilde{u}^m(\cdot,t)|^2 + r(0) \cdot \int_0^t\int_\Omega |\partial_s \tilde{u}^m(\cdot,s)|^2 ds + \int_0^t\int_\Omega \partial_s \tilde{u}^m(\cdot,s)\cdot(\dot{r} * \partial_s \tilde{u}^m)(\cdot,s)ds$$

$$\geq \frac{1}{2}\int_\Omega |\partial_t \tilde{u}^m(\cdot,t)|^2 - C \cdot \int_0^t\int_\Omega |\partial_s \tilde{u}^m(\cdot,s)|^2 ds ,$$

$$I_2(t) = \int_0^t\int_\Omega |\nabla_x\partial_s \tilde{u}^m(\cdot,s)|^2 ds + \frac{r(0)}{2} \cdot \int_\Omega |\nabla_x \tilde{u}^m(\cdot,t)|^2 + \int_0^t\int_\Omega \nabla_x\partial_s \tilde{u}^m(\cdot,s)\cdot(\dot{r} * \nabla_x \tilde{u}^m)(\cdot,s)ds$$

$$\geq \frac{1}{2}\int_0^t\int_\Omega |\nabla_x\partial_s \tilde{u}^m(\cdot,s)|^2 ds + \frac{r(0)}{2}\int_\Omega |\nabla_x \tilde{u}^m(\cdot,t)|^2 - C \cdot \int_0^t\int_\Omega |\nabla_x \tilde{u}^m(\cdot,s)|^2 ds ,$$

$$I_3(t) = \int_0^t\int_\Omega \nabla_x\partial_s u^m(\cdot,s)\cdot g(\nabla_x u^m(\cdot,s))ds = \int_\Omega \tilde{G}(\nabla_x u^m(\cdot,t)) - \frac{L}{2} \cdot (\int_\Omega |\nabla_x u^m(\cdot,t)|^2 + 1) ,$$

$$I_4(t) = \int_0^t\int_\Omega (\partial_s f(\cdot,s) + r * \partial_s f(\cdot,s) - \partial_s^2 u_0(\cdot,s) - r * \partial_s^2 u_0(\cdot,s))\cdot\partial_s \tilde{u}^m(\cdot,s)ds$$

$$- \int_0^t\int_\Omega (\nabla_x\partial_s u_0(\cdot,s) + r * \nabla_x\partial_s u_0(\cdot,s)) \cdot \nabla_x\partial_s \tilde{u}^m(\cdot,s)ds$$

$$+ \int_0^t\int_\Omega \nabla_x\partial_s u_0(\cdot,s) \cdot g(\nabla_x u^m(\cdot,s))ds$$

$$\leq \int_0^t \bar{C}(s)\cdot(\int_\Omega |\partial_s \tilde{u}^m(\cdot,s)|^2)^{\frac{1}{2}} ds + \frac{1}{4}\int_0^t\int_\Omega |\nabla_x\partial_s \tilde{u}^m(\cdot,s)|^2 ds + C\cdot\int_0^t\int_\Omega \tilde{G}(\nabla_x u^m(\cdot,s))ds + C ,$$

with $\tilde{C} \in L^1(0,T;\mathbb{R})$; using the properties of g, u_0, and f in the last estimate. Inserting all these estimates into (2.7) and using Gronwall's lemma we get (2.5).

Step 3: We extract a subsequence of the $(u^m)_{m\geq 1}$, again labeled in the same way, such that

(i) $u^m \to u$ strongly in $L^2(0,T;L^2(\Omega))$;

(ii) $\nabla_x u^m \to \nabla_x u$ weakly in $L^2(0,T;L^2(\Omega))$;

(iii) $\partial_t u^m \to \partial_t u$ weakly in $L^2(0,T;W^{1,2}(\Omega))$;

(iv) $g(\nabla_x u^m) \to \zeta$ weakly in $L^2(0,T;L^1(\Omega,\mathbb{R}^n))$,

with a suitable function ζ. All these limits exist due to suitable imbedding theorems; the choice (iv) is possible since the $g(\nabla u^m)$ are equi-integrable and hence weakly sequentially precompact in $L^1([0,T] \times \Omega,\mathbb{R}^n)$ (cf. [9] and Lemmas 5.3, 5.4).

Next we want to use that actually

(v) $\partial_t u^m \to \partial_t u$ strongly in $L^2(0,T;L^2(\Omega))$.

Suppose this is true; what is needed to complete the proof of the theorem now is

$$\xi = g(\nabla_x u) \quad \text{a.e. on } \Omega \times [0,T] . \tag{2.8}$$

To show this, we use a version of a monotonicity argument which has first been employed by Clements ([4]). Transform (2.4) again by means of (2.6), differentiate the resulting identity once, and take a test function $\partial_\tau v$, $v \in W^{1,2}([0,T],V_m)$. Integrating the resulting identity from 0 to s with respect to t and from 0 to T with respect to s then gives

$$\begin{aligned} -\int_0^T (T-t)\int_\Omega (\partial_t u^m \partial_t v)dt + \int_0^T\int_\Omega \partial_t u^m \cdot v\, dt - T\int_\Omega \partial_t u^m(\cdot,0)\cdot v(\cdot,0) \\ + \int_0^T (I_{m,1}(v)(t) + I_{m,2}(v)(t) + I_{m,3}(v)(t))dt = \int_0^T I_4(v)(t)dt , \end{aligned} \tag{2.9}$$

using the abbreviations

$$I_{m,1}(v)(t) = (T-t)\int_\Omega (r * \partial_t u^m)(\cdot,t)\cdot v(\cdot,t)dt\ ,$$

$$I_{m,2}(v)(t) = \int_0^t\int_\Omega \nabla_x\partial_t u^m(\cdot,s)\nabla_x v(\cdot,s)ds + (T-t)\int_\Omega \{(r(0)-L)\nabla_x u^m(\cdot,t) - r(t)\nabla_x u^m(\cdot,0)$$

$$+ \dot{r} * \nabla_x u^m(\cdot,t)\}\nabla_x v(\cdot,t)\ ,$$

$$I_{m,3}(v)(t) = (T-t)\int_\Omega \{g(\nabla_x u^m(\cdot,t)) + L\cdot\nabla_x u^m(\cdot,t)\}\cdot\nabla_x v(\cdot,t)\ ,$$

$$I_4(v)(t) = (T-t)\int_\Omega \{\partial_t f(\cdot,t) + r * \partial_t f(\cdot,t)\}\cdot v(\cdot,t)\ .$$

As $m\to\infty$, we can replace $I_{m,k}(v)(\cdot,t)$ by $I_k(v)(\cdot,t)$ $(1\le k\le 3)$ and $\partial_t u^m$ by $\partial_t u$, $\nabla\partial_t u^m$ by $\nabla\partial_t u$, $g(\nabla u^m)$ by ζ, in obvious notation, using (i)-(iv). The resulting identity then holds for any $v\in W^{1,2}([0,T],W_0^{1,\infty}(\Omega))$ (by density). More precisely, we only need v to be in $W^{1,2}([0,T],L^2(\Omega)),L^2(0,T;W_0^{1,2}(\Omega))$ and additionally

$$\sup_{[0,T]}\int_\Omega \tilde{G}(\nabla_x v(\cdot,t)) < \infty\ ,$$

as is shown in Lemma 5.5. We now insert $v(\cdot,t) = e^{-\alpha t}\cdot(u(\cdot,t)-u_0(\cdot,t))$, $\alpha>0$ to be chosen later. Writing $u_\alpha(\cdot,t) = e^{-\alpha t}u(\cdot,t)$, $u_{0,\alpha}(\cdot,t) = e^{-\alpha t}u_0(\cdot,t)$, we find the identity

$$(2.10)\quad -\int_0^T (T-t)e^{-\alpha t}\int_\Omega |\partial_t u(\cdot,t)|^2dt + \int_0^T (\alpha(T-t)+1)e^{-\alpha t}\cdot\int_\Omega \partial_t u(\cdot,t)\cdot u(\cdot,t)dt$$

$$+\int_0^T e^{-\alpha t}\cdot[\{\tfrac{1}{2} + (T-t)(r(0)-L+\tfrac{\alpha}{2})\}\int_\Omega |\nabla_x u(\cdot,t)|^2 + \int_\Omega \dot{r}*\nabla_x u(\cdot,t)\cdot\nabla_x u(\cdot,t)]dt$$

$$+\int_0^T (I_1(u_\alpha)(t) + I_3(u_\alpha)(t))dt = \int_0^T (I_4(u_\alpha)(t) + I_1(u_{0,\alpha})(t) + I_2(u_{0,\alpha})(t)$$

$$+ I_3(u_{0,\alpha})(t))dt + \int_0^T\int_\Omega \partial_t u(\cdot,t)\cdot\{\partial_t u_0(\cdot,t)e^{-\alpha t}(t-T) + (\alpha(T-t)+1)e^{-\alpha t}u_0(\cdot,t)\}dt$$

$$+\frac{T}{2}\cdot\int_\Omega |\nabla_x u_0(\cdot,0)|^2 + \int_0^T (T-t)r(t)e^{-\alpha t}\int_\Omega \nabla_x u_0(\cdot,0)\nabla_x u(\cdot,t)dt\ .$$

Next, we insert $v(\cdot,t) = e^{-\alpha t} \cdot (u^m(\cdot,t) - u_0(\cdot,t))$ in (2.9). After rearranging we get

$$(2.11) \qquad - \int_0^T (T-t)e^{-\alpha t} \int_\Omega |\partial_t u^m(\cdot,t)|^2 \qquad \int_0^T (\alpha(T-t)+1)e^{-\alpha t} \int_\Omega \partial_t u^m(\cdot,t)u^m(\cdot,t)dt +$$

$$+ \int_0^T e^{-\alpha t}[\{\tfrac{1}{2} + (T-t)(r(0)-L + \tfrac{\alpha}{2})\} \int_\Omega |\nabla_x u^m(\cdot,t)|^2 + \int_\Omega \mathring{r}*\nabla_x u^m(\cdot,t)\nabla_x u^m(\cdot,t)]dt$$

$$+ \int_0^T \int_\Omega (g(\nabla_x u^m) + L \cdot \nabla_x u^m) \cdot \nabla_x u^m \cdot e^{-\alpha t}(T - t)dt = C(m) ,$$

$C(m)$ contains only terms which have corresponding expressions in (2.10) as limits. We take α big enough such that the form

$$(2.12) \qquad v \mapsto \int_0^T e^{-\alpha t}[\{\tfrac{1}{2} + (T-t)(r(0)-L + \tfrac{\alpha}{2})\}\cdot v^2(t) + \mathring{r}*v(t)\cdot v(t)]dt \quad \text{for} \quad v \in L^2(0,T;\mathbb{R})$$

is positive definite. Now take the lim inf in (2.11) as $m \to \infty$. The first two integrals on the left hand side converge due to (v); the third integral is the positive definite form that appears in (2.12) and is hence lower semi-continuous with respect to weak convergence. Comparing the result with (2.10) we see that

$$\liminf_{m\to\infty} \int_0^T e^{-\alpha t}(T - t) \int_\Omega (g(\nabla_x u^m(\cdot,t)) + L \cdot \nabla_x u^m(\cdot,t)) \cdot \nabla_x u^m(\cdot,t)dt <$$

$$< \int_0^T e^{-\alpha t}(T - t) \int_\Omega (\zeta(\cdot,t) + L \cdot \nabla_x u(\cdot,t)) \cdot \nabla_x u(\cdot,t)dt .$$

Then a standard argument using the monotonicity of $p \mapsto g(p) + L \cdot p$ implies that $g(\nabla_x u) = \zeta$ a.e. on $\Omega \times [0,T]$ (cf. [15]).

It remains to be shown that (v) holds. Let $w_m = \partial_t u^m \in W^{1,\infty}([0,T],V_m)$. We use a version of a compactness argument in [15] to show that $(w_m)_{m\in\mathbb{N}}$ is a Cauchy-sequence in $L^2(0,T;L^2(\Omega))$, from which (v) follows.

First, let $X_m = V_m$, equipped with the $W^{1,\infty}$-norm, X be the $W^{1,\infty}$-closure of $\bigcup_{m=1}^{\infty} X_m$, and X^*, X_m^* be the corresponding dual spaces. We claim:

For any $\varepsilon > 0$ there exist $C(\varepsilon) > 0$ and $K \in \mathbb{N}$ such that for all $z \in W^{1,2}(\Omega)$

$$\|z\|_{L^2(\Omega)} \le \varepsilon \cdot \|z\|_{W^{1,2}(\Omega)} + C(\varepsilon) \cdot \|z\|_{X_K^*} . \tag{2.13}$$

For else we could find an $\varepsilon > 0$ and a sequence $(z_K)_{K\geq 1}$ in $W^{1,2}(\Omega)$, $\|z_K\|_{W^{1,2}} = 1$, such that

$$\|z_K\|_{L^2(\Omega)} > \varepsilon + K \cdot \|z_K\|_{X_K^*} . \tag{2.14}$$

Using the compactness of the imbedding $W^{1,2}(\Omega) \to L^2(\Omega)$ we extract a subsequence with L^2-limit $\bar{z}, \|\bar{z}\|_{L^2(\Omega)} > \varepsilon$. On the other hand, by (2.14) $\|z_K\|_{X_K^*} \to 0$.

By density, this implies that $z_K \to 0$ weak-* in X^*, which is a contradiction to $z_{\ell(k)} \to \bar{z} \neq 0$ in $L^2(\Omega)$ for a suitable subsequence. From (2.13) we conclude that for all $\varepsilon > 0$ there exist C, K such that for all $w \in L^2(0,T;W^{1,2}(\Omega))$

$$\|w\|_{L^2(0,T;L^2(\Omega))} \le \varepsilon \cdot \|w\|_{L^2(0,T;W^{1,2}(\Omega))} + C \cdot \|w\|_{L^2(0,T;X_K^*)} . \tag{2.15}$$

We apply (2.15) to $(w_m)_{m\geq 1}$ and see that it suffices to show that this is a Cauchy-sequence in any $L^2(0,T;X_K^*)$. In fact, since $w_m \longrightarrow \partial_t u$ weakly in $L^2(0,T;L^2(\Omega))$ by (iv), it will be enough to show that $(w_m)_{m\geq 1}$ is precompact in any $L^2(0,T;X_K^*)$; the claim then follows from standard diagonal sequence arguments. Now the differentiated version of (2.4) shows that for fixed K

$$\{\|\partial_t w_m\|_{X_K^*} \mid K \le m\}$$

is an equi-integrable set in $L^1(0,T;\mathbb{R})$. Hence for fixed K the w_m are equi-continuous in X_K^* and by (2.14)

$$\langle w_m(0), v\rangle_{X_K, X_K^*} = \int_\Omega (-\nabla_x u_0 \cdot \nabla_x v + f \cdot v) ,$$

which shows that $(w_m(0))_{m\geq 1}$ is uniformly bounded in X_K^*. Arzela's theorem then implies that $(w_m)_{m\geq 1}$ is precompact even in $C([0,T],X_K^*)$ for any K.

This argument completes the proof of (v) and thus of the theorem.

Remarks

2.2. As in, e.g., [6], it is possible to weaken the assumptions on a to

$$a \in W^{1,\infty}([0,T],\mathbb{R}),\ \dot{a} \in BV([0,T],\mathbb{R}),\ a(0) = 1\ ,$$

and the proof of Theorem 2.1 even allows to include x-dependent a, e.g.

$$a \in W^{2,1}([0,T],L^{\infty}(\Omega)),\ a(\cdot,0) \equiv 1\ .$$

2.3. The condition (A2) basically requires g to be of "polynomial" character such that (2.1) holds. However, g can be "anisotropic" in the sense that it can possess different growth properties along different directions in $\mathbb{R}^n$. Also, g can be "degenerate"; e.g.,

$$g(\xi) = (1 + |\xi|^2)^{-\alpha} \cdot \xi,\ \alpha > 0$$

is allowed, and g can be "non-monotone" (only $\xi \mapsto g(\xi) + L \cdot \xi$ has to be monotone); cf. [1] for an even weaker assumption in the case of one space dimension. Finally, the proof allows also to include x-dependent g's or an elliptic differential operator in divergence form instead of the Laplacian.

3. Differentiable Solutions

In this section we want to give a somewhat different existence argument for solutions of (I). It is partly based on a method that was used in [2] and [6] to treat a Hilbert-space version of (I) and will enable us to include the nonlinear boundary condition (1.2) and to show that all terms in (I) actually exist as locally integrable functions. On the other hand, we shall only treat the "isotropic" case $g(p) = g_0(|p|) \cdot p$. Some variants of the assumptions made above will be used:

(B1) The region $\Omega \subset \mathbb{R}^n$ is open and bounded. $\partial\Omega$ is a Lipschitz manifold, $\Gamma \subset \partial\Omega$ is a submanifold of dimension $n-1$, Ω is locally on one side of Γ.

(B2) The function $g : \mathbb{R}^n \to \mathbb{R}^n$ is given by $g(p) = g_0(|p|) \cdot p$, $g_0 : [0,\infty) \to \mathbb{R}$ being locally Lipschitz-continuous on $(0,\infty)$. There exist constants $L_1 > 0$, $C > 0$, $\delta > 0$, such that for all $r > 0$

$$g_0(r) + L_1 > 0 \tag{3.1}$$

$$\delta \cdot (g_0(r) + L_1) \le \frac{d}{dr}((g_0(r) + L_1) \cdot r) \le C \cdot (g_0(r) + L_1) . \tag{3.2}$$

Similar to Section 2, we define $G_0(r) = \int_0^r g_0(s) \cdot s\,ds$ for $r > 0$, $G(p) = G_0(|p|)$, and $\tilde{G}(p) = G(p) + \frac{1}{2} L_1 \cdot |p|^2$.

(B3) The function $\beta : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous, and there exists a constant $L_2 > 0$ and for any $\varepsilon > 0$ a $C(\varepsilon) > 0$ such that for all $r \in \mathbb{R}$

$$-L_2 \le \beta'(r) \le \varepsilon \cdot (\beta(r) + L_2 \cdot r) + C(\varepsilon) . \tag{3.3}$$

Without loss of generality, $L_1 = L_2 = L > 0$. Define

$$B(r) = \int_0^r \sqrt{\beta'(s) + L}\, ds .$$

(B4) The function u_0 satisfies

$$\int_\Gamma B^2(u_0(\cdot,0)) < \infty, \quad \sup_{\Gamma\times[0,T]} |\partial_t u_0| < \infty .$$

The main result of this section is

Theorem 3.1: Suppose that (B1)-(B4), (A3)-(A5) hold. Then the equation (I) together with initial and boundary conditions (1.0)-(1.2) has a distributional solution u; i.e. u satisfies

$$\int_0^T\int_\Omega \{\nabla_x u + a * g(\nabla_x u)) \cdot \nabla_x\phi - u \cdot \partial_t\phi - f \cdot \phi\}dxdt + \int_0^t\int_\Gamma w \cdot \phi dxdt$$

$$= \int_\Omega u_0(\cdot,0) \cdot \phi(\cdot,0)dx$$

for all test functions $\phi \in C_0^1((\Omega \cup \Gamma) \times [0,T);\mathbb{R})$; $w \in L^1(\Gamma \times [0,T],\mathbb{R})$, such that $w = \beta(u)$ a.e. on $\Gamma \times [0,T]$; $(u - u_0)(\cdot,t)$ is for a.e. t in the $W^{1,2}$-closure of $C_0^1((\Omega \cup \Gamma),\mathbb{R})$.

Also,

$$\int_0^T \{\int_\Omega |\partial_t\nabla_x u|^2 + \int_\Gamma |\partial_t\beta(u)|^2\}dt + \sup_{[0,T]} \int_\Omega (|\partial_t u(\cdot,t)|^2 + G(\nabla_x u)) < \infty, \tag{3.4}$$

and

$$\int_0^T\int_\Omega \xi^2 \cdot |\nabla_x(\sqrt{g_0(|\nabla_x u|) + L} \cdot \nabla_x u)|^2 dxdt + \sup_{[0,T]} \int_\Omega \xi^2 \cdot |\nabla_x^2 u|^2 < \infty, \tag{3.5}$$

if $\xi : \Omega \to \mathbb{R}$ is Lipschitz-continuous, $\xi|_{\partial\Omega} \equiv 0$, $\|\nabla_x\xi\|_{L^\infty} \leq 1$ (e.g., if $\xi(x) = \mathrm{dist}(x,\partial\Omega)$).

Proof. We shall

(i) find solutions of approximating equations,

(ii) derive the estimates (3.4) and (3.5) and

(iii) pass to the limit.

For $M > 0$ we define

$$g_0^M(r) = \inf(g_0(r),M), \qquad g^M(p) = g_0^M(|p|) \cdot p\ ;$$

$$\beta^M(r) = \inf(M,\sup(\beta(r),-M))\ .$$

Clearly, g_0^M and β^M fulfill (B2) and (B3) with the same constants.

Step 1: We solve (I) with g replaced by g^M, β replaced by β^M, and get solution u^M by means of a Galerkin-type argument similar to the one used in the previous section (cf. [11] for an abstract existence theory for similar problems). The u^M are unique, since g^M and β^M are globally Lipschitz-continuous. Define

$$B^M(r) = \int_0^r \sqrt{\beta^{M'}(s) + L}\, ds ,$$

$$G^M(p) = \int_0^{|p|} g_0^M(s) \cdot s ds .$$

We show that (3.4) still holds with G, B, u replaced by G^M, B^M, u^M, the bound not depending on M. To this end we take backward difference quotients in (I), use the backward difference quotient $d_h(u^M - u_0)$ as a test function (which is admissible), integrate over $[h,t]$, and let h tend to zero. The result is the following identity:

$$\int_0^t \int_\Omega \partial_s \nabla_x u^M \cdot \partial_s \nabla_x (u^M - u_0) dxdt + \int_0^t \int_\Omega (g(\nabla_x u^M) + \dot{a} * g(\nabla_x u^M)) \cdot \partial_s \nabla_x (u^M - u_0) dxdt$$

$$(3.6) \qquad + \int_\Omega \partial_s u^M(\cdot,t) \cdot \partial_s (u^M - u_0)(\cdot,t) dx + \int_0^t \int_\Gamma \partial_s \beta^M(u^M) \cdot \partial_s (u^M - u_0) dxdt =$$

$$= \int_0^t \int_\Omega \partial_s f \cdot \partial_s (u^M - u_0) dxdt .$$

The manipulations that lead to the estimates of $\partial_t \nabla_x u^M$ and $\partial_t u^M$ are the same as in the corresponding part of the proof of Theorem 2.1. Only the two integrals that contain the nonlinear terms $g^M(\nabla_x u^M)$ and $\beta^M(u^M)$ need some additional arguments:

$$\int_0^t\int_\Omega \partial_s\nabla_x u^M(\cdot,s)\cdot(g^M(\nabla_x u^M)+\dot{a}*g^M(\nabla_x u^M))(\cdot,s)ds =$$

$$= \int_\Omega G^M(\nabla_x u^M(\cdot,t)) - \int_\Omega G^M(\nabla_x u^M(\cdot,0)) + \int_\Omega \nabla_x u^M(\cdot,t)\cdot\dot{a}*g^M(\nabla_x u^M)(\cdot,t)$$

$$- \int_0^t\int_\Omega \dot{a}(0)\cdot\nabla_x u^M(\cdot,s)\cdot g^M(\nabla_x u^M(\cdot,s))ds - \int_0^t\int_\Omega \nabla_x u^M(\cdot,s)\cdot\ddot{a}*g^M(\nabla_x u^M(\cdot,s))ds ,$$

where we have performed an integration by parts.

We use Lemma 5.6 to estimate this from below by

$$\frac{1}{2}\int_\Omega G^M(\nabla_x u^M(\cdot,t)) - \frac{1}{2}\int_\Omega |\nabla_x u^M(\cdot,t)|^2 - C\cdot(\int_0^t\int_\Omega (G^M(\nabla_x u^M(\cdot,s))$$

$$+ L|\nabla_x u^M(\cdot,s)|^2)ds + 1) .$$

Also,

$$|\int_0^t\int_\Omega \nabla_x\partial_s u_0(\cdot,s)\cdot\partial_s(a*g^M(\nabla_x u^M)(\cdot,s))ds| \le$$

$$\le C\cdot\int_0^t\int_\Omega (G(\nabla_x\partial_s u_0(\cdot,s)) + L\cdot|\nabla_x\partial_s u_0(\cdot,s)|^2 + G^M(\nabla_x u^M(\cdot,s)) + L\cdot|\nabla_x u^M(\cdot,s)|^2)ds$$

by the same lemma.

Concerning the other nonlinear term, we have $\partial_t u^M \in L^2(\Gamma\times[0,T])$ by Lemma 5.1 and

$$\int_0^t\int_\Gamma \partial_s\beta^M(u^M(\cdot,s))\cdot\partial_s(u^M-u_0)(\cdot,s)ds$$

$$= \int_0^t\int_\Gamma |\partial_s\beta^M(u^M(\cdot,s))|^2 ds - L\cdot\int_0^t\int_\Gamma |\partial_s u^M(\cdot,s)|^2 - \int_0^t\int_\Gamma \beta^M(u^M)\cdot\partial_s u^M(\cdot,s)\cdot\partial_s u_0(\cdot,s)ds .$$

We use Lemmas 5.1 and 5.2 to estimate this from below by

$$\frac{1}{2}\int_0^t\int_\Gamma |\partial_s\beta^M(u^M(\cdot,s))|^2 ds - \varepsilon\cdot\int_0^t\int_\Omega |\nabla_x\partial_s u^M(\cdot,s)|^2 ds - C(\varepsilon)\cdot\int_0^t\int_\Omega |\partial_s u^M(\cdot,s)|^2 ds$$

$$- C\cdot\left(\int_0^t\int_\Gamma |\beta^M(u^M(\cdot,s))|^2 ds + 1\right) .$$

Collecting all terms we then get (3.4) after an application of Gronwall's Lemma.

<u>Step 2</u>: We want to use $\xi^2\cdot\Delta u^M$ as a test function, $\xi\in W_0^{1,\infty}(\Omega)$, and hence have to show that $\xi\cdot\nabla_x^2 u^M$ is in $L^2(\Omega\times[0,T])$ for such ξ. To this end we replace the kernel a by $a^\varepsilon(\cdot) = 1_{[\varepsilon,\infty)}(\cdot)\cdot a(\cdot)$ and consider the elliptic problems

(3.7) $$-\Delta u^{M,\varepsilon}(\cdot,t) - \operatorname{div}(a^\varepsilon * g^M(\nabla_x u^{M,\varepsilon}))(\cdot,t) = f(\cdot,t) - \partial_t u^M(\cdot,t)$$

for $0 \le t \le T$ with corresponding boundary conditions. The problems (3.7) can be solved step by step, its solutions $u^{M,\varepsilon}$ satisfy

(3.8) $$\sup_{[0,T]}\int_\Omega \xi^2\cdot|\nabla_x^2 u^{M,\varepsilon}|^2(\cdot,t) \le K(M) \quad \text{for all } \varepsilon \text{ and } M ,$$

$K(M)$ independent of ε, if we use the estimate (3.4) for $\partial_t u^M$, the global Lipschitz condition for g^M and β^M, and standard results for linear elliptic equations ([14]). Passing to the limit as $\varepsilon\downarrow 0$ we recover the u^M for which thus (3.8) still holds. Hence (I) holds a.e. in $\Omega\times[0,T]$, and all the summands appearing in (I) are in $L^2(\Omega_1\times[0,T])$ for any compact subdomain $\Omega_1\subset\subset\Omega$. Let again r denote the resolvent kernel of $\dot a$:

$$r(t) + \int_0^t \dot a(t-s)r(s)ds + \dot a(t) = 0 \quad \text{on } [0,T] .$$

Taking the convolution of (I) with r and adding it to (I) then gives

(3.9) $$\partial_t u^M(\cdot,t) + r*(\partial_t u^M)(\cdot,t) - \Delta_x u^M(\cdot,t) - r*\Delta_x u^M(\cdot,t) - \\ - 1*\operatorname{div} g^M(\nabla_x u^M)(\cdot,t) = f(\cdot,t) + r*f(\cdot,t) .$$

Let again d_h denote the backward difference quotient, and let $\xi\in W^{1,\infty}(\Omega)$, $\operatorname{supp}\xi = \Omega_1\subset\subset\Omega$, $\|\nabla\xi\|_{L^\infty}\le 1$.

We apply d_h to (3.9) in $h \le s \le T$, multiply with $\xi^2 \cdot (-\Delta_x u^M(\cdot,s))$, and integrate over $\Omega \times [h,t]$, $t \le T$. As $h \downarrow 0$, all limits exist a.e., and we get the identity (for a.e. t)

$$(3.10)\qquad -\int_\Omega \partial_t u^M(\cdot,t)\cdot\Delta_x u^M(\cdot,t)\cdot\xi^2 - \int_0^t\int_\Omega \{|\nabla_x\partial_t u^M|^2\cdot\xi^2 + 2\xi(\nabla_x\xi\cdot\nabla_x\partial_t u^M)\cdot\partial_t u^M\}$$

$$+\int_0^t\int_\Omega r(0)\partial_t u^M\cdot(-\Delta_x u^M)\cdot\xi^2 + \int_0^t\int_\Omega (\dot r * \partial_t u^M)\cdot(-\Delta_x u^M)\cdot\xi^2 +$$

$$+\frac{1}{2}\int_\Omega \xi^2\cdot|\Delta_x u^M(\cdot,t)|^2 + \int_0^t\int_\Omega \{r(0)\Delta_x u^M + \dot r * \Delta_x u^M\}\cdot\Delta_x u^M\cdot\xi^2$$

$$+\int_0^t\int_\Omega \Delta_x u^M\cdot\mathrm{div}_x(g^M(\nabla_x u^M))\cdot\xi^2 + \int_\Omega \{f(\cdot,0) + \frac{1}{2}\Delta_x u_0(\cdot,0)\}\cdot\Delta_x u_0(\cdot,0)\cdot\xi^2$$

$$= \int_0^t\int_\Omega \{\partial_t f + r(0)f + \dot r * f\}\cdot(-\Delta_x u^M)\cdot\xi^2 .$$

Rearranging this and using the estimates for $\partial_t u^M$ and $\nabla_x\partial_t u^M$ we get for a.e. t

$$(3.11)\qquad \frac{1}{4}\int_\Omega |\Delta_x u^M|^2\cdot\xi^2 + \int_0^t\int_\Omega \Delta_x u^M\cdot\mathrm{div}_x(g^M(\nabla_x u^M) + L\cdot\nabla_x u^M)\cdot\xi^2 \le$$

$$\le C_1 + \int_0^t C_2(s)\int_\Omega |\Delta_x u^M|^2(\cdot,s)\xi^2\,dx\,ds$$

with some $C_1 > 0$, $C_2 \in L^1(0,T;\mathbb{R})$ independent of M,L as in (B2) and big enough. We want to estimate the second integral on the left hand side: Fix M and $0 \le s \le t$ and write $\tilde g(r) = g_0^M(r) + L$ for short. Then (suppressing s-dependence and writing ∂_i for ∂_{x_i})

$$(3.12)\qquad \int_\Omega \Delta_x u^M\cdot\mathrm{div}_x(\tilde g(|\nabla_x u^M|)\cdot\nabla_x u^M)\cdot\xi^2 =$$

$$= \sum_{i,j}\int_\Omega \partial_i\partial_j u^M\cdot\partial_j(\tilde g(|\nabla_x u^M|)\cdot\partial_i u^M)\cdot\xi^2 +$$

$$+\sum_{i,j}\int_\Omega 2\cdot\xi\cdot\partial_i\xi\cdot(\partial_j\partial_j u^M\cdot\tilde g(|\nabla_x u^M|)\cdot\partial_i u^M - \partial_i\partial_j u^M\tilde g(|\nabla u^M|)\cdot\partial_j u^M) .$$

Now for a.e. $x \in \Omega$

$$\sum_{i,j} \partial_i\partial_j u^M \cdot \partial_j((\tilde{g}(|\nabla u^M|)\partial_i u^M) =$$

$$= \kappa \cdot \sum_{i,j} |\partial_i(\sqrt{\tilde{g}(|\nabla u^M|)}\ \partial_j u^M)|^2 + (1-\kappa)\tilde{g}(|\nabla u^M|) \sum_{i,j} |\partial_i\partial_j u^M|^2 +$$

$$+ ((1-\kappa)\cdot\tilde{g}'(|\nabla u^M|)\cdot|\nabla u^M| - \kappa \cdot \frac{(\tilde{g}'(|\nabla u^M|))^2}{4\tilde{g}(|\nabla u^M|)} \cdot |\nabla u^M|^2)\cdot \sum_{i,j} |\partial_i\partial_j u^M \partial_i u^M|^2 \cdot \frac{1}{|\nabla u^M|^2},$$

$\kappa > 0$ to be chosen later. Writing $p = |\nabla u^M|$ and $d = \sum_{i,j} |\partial_i\partial_j u^M|^2$ for short, we estimate further

$$\ldots \geq \kappa \cdot \sum_{i,j} |\partial_i(\sqrt{\tilde{g}(|\nabla u^M|)}\ \partial_j u^M)|^2 + \left((1-\kappa)\cdot\delta\cdot\tilde{g}(p) - \kappa \cdot \frac{(\tilde{g}'(p))^2\cdot p^2}{4\tilde{g}(p)}\right)\cdot d,$$

δ as in (3.2). Choosing κ small enough we see from (3.2) that this expression is bounded from below by

$$\kappa \cdot \sum_{i,j} |\partial_i(\sqrt{\tilde{g}(|\nabla u^M|)}\ \partial_j u^M|^2 + \frac{\delta}{2}\cdot\tilde{g}(p)\cdot d .$$

This term, multiplied with ξ^2 and integrated over Ω, hence gives a lower bound for the first integral on the right hand side of (3.12).

We estimate the second integral, using the same notation:

$$\sum_{i,j} \int_\Omega 2\xi\partial_i\xi(\partial_j\partial_j u^M \cdot \tilde{g}(p) \cdot \partial_i u^M - \partial_i\partial_j u^M \tilde{g}(p) \cdot \partial_j u^M)$$

$$\leq \frac{\delta}{2}\cdot\int_\Omega \xi^2\cdot\tilde{g}(p)\cdot d + C(\delta)\cdot\int_\Omega \tilde{g}(p)\cdot p^2,$$

δ as in (3.2). But the second integral can be estimated by $\int_\Omega \tilde{G}^M(\nabla_x u^M)$ and $\int_\Omega |\nabla_x u^M|^2$ (Lemma 5.6), and this term is bounded by the estimate (3.4) uniformly in t and M. Hence from (3.11) we get

$$(3.13)\qquad \frac{1}{4}\int_{\Omega} |\Delta_x u^M|^2 \cdot \xi^2 + \kappa \cdot \int_0^t\int_{\Omega} \sum_{i,j} |\partial_i(\sqrt{g_0^M(|\nabla u^M|) + L}\ \partial_j u^M)|^2 \cdot \xi^2$$

$$\leq C_1 + \int_0^t C_2(s) \int_{\Omega} |\Delta_x u^M(\cdot,s)|^2 \cdot \xi^2 ds ,$$

and Gronwall's Lemma implies

$$(3.14)\qquad \sup_{[0,T]} \int_{\Omega} |\Delta_x u^M|^2 \cdot \xi^2 + \int_0^T\int_{\Omega} \sum_{i,j} |\partial_i(\sqrt{g_0^M(|\nabla u^M|) + L}\ \partial_j u^M)|^2 \cdot \xi^2 \leq K ,$$

K not depending on M.

<u>Step 3</u>: We extract a subsequence of the $(u^M)_{M\geq 1}$ (not relabelled) such that

(i) $u^M \to u$ strongly in $L^2(0,T;L^2(\Omega))$,

strongly in $L^2(0,T;L^2(\Gamma))$, and a.e. on $\Gamma \times [0,T]$;

(ii) $\partial_t u^M \to \partial_t u$ weakly in $L^2(0,T;W^{1,2}(\Omega))$;

(iii) $\nabla_x u^M \to \nabla_x u$ strongly in each $L^2(0,T;L^2(\Omega'))$, $\Omega' \subset\subset \Omega$.

Also, the estimate (3.4) together with the properties of β and g_0 (cf. Lemma 5.4) shows that $\beta^M(u^M)$ and $g_0^M(|\nabla_x u^M|) \cdot \nabla_x u^M$ are equi-integrable families and hence weakly precompact in $L^1(0,T;L^1(\Gamma))$ resp. in $L^1(0,T;L^1(\Omega))$. Hence we can choose the subsequence such that

(iv) $\beta^M(u^M) \to \eta$ weakly in $L^1(0,T;L^1(\Gamma))$;

(v) $g_0^M(|\nabla_x u^M|)\nabla_x u^M \to \xi$ weakly in $L^1(0,T;L^1(\Omega))$.

The continuity of β and g_0 together with (i) and (iii) then show that

$$\eta = \beta(u) \text{ a.e. on } \Gamma \times [0,T]$$

$$\xi = g_0(|\nabla u|) \cdot \nabla u \text{ a.e. on } \Omega \times [0,T] ,$$

and moreover (3.14) still holds for the limit function u. Hence u solves (I), and $\nabla_x^2 u$ and $a * \operatorname{div}(g_0(|\nabla_x u|) \cdot \nabla_x u)$ are in $L^\infty(0,T;L^2_{loc}(\Omega))$ resp. in $L^\infty(0,T;L^1_{loc}(\Omega))$. Theorem 3.1 is proved.

Remarks.

3.2. Since no differentiability properties of u_0 on $\partial\Omega\setminus\Gamma$ are ever used, one can weaken (B1) to the following hypothesis:

There exist $(n-1)$-dimensional Lipschitz-manifolds Γ, Γ_0, such that $\Gamma \subset \mathring{\Gamma}_0 \subset \Gamma_0 \subset \partial\Omega$, and Ω is locally on one side of Γ_0.

Since u_0 determines the behavior of solutions only on $\partial\Omega\setminus\Gamma$, (B4) then is only a condition for u_0 on $(\Gamma_0\setminus\Gamma) \times [0,T]$ (by suitable extension arguments.). Also, no additional problems arise if one replaces $\beta(u(x))$ by $\beta(u(x)) + h(x,t)$, $h : \Gamma \times [0,T] \to \mathbb{R}$ in a suitable Sobolev class.

3.3. It is possible to take $\beta = \beta_0 + \beta_1$ as a boundary nonlinearity, β_0 as in (B3), β_1 being maximal monotone and sublinear, at the expense of assuming more regularity properties for $u_0(\partial_t^2 u_0 \in L^1(0,T;L^2(\Gamma)))$. Also, if $\tilde{G}(\xi) \geq \varepsilon \cdot |\xi|^p - C$, $\varepsilon > 0$, $p > n$, then β need only be continuous and $\beta'(r) \geq -L_2$, since then the approximating solutions will converge uniformly on $\Gamma \times [0,T]$.

3.4. It should be possible to extend the class of functions $g : \mathbb{R}^n \to \mathbb{R}^n$ to "anisotropic" ones (the Jacobians Dg having isotropic spectral radii, however), satisfying, e.g.:

There exist $\varepsilon, K_1, K_2 > 0$ and $\mu_*, \mu^* : [0,\infty) \to \mathbb{R}$ such that for all $\xi, p \in \mathbb{R}^n$

$$(\mu_*(|p|) - K_1) \cdot |\xi|^2 \leq \xi^T \cdot Dg(p) \cdot \xi \leq (\mu^*(|p|) + K_1) \cdot |\xi|^2 ,$$

$$\mu_*(|p|) \cdot |p|^2 \geq \varepsilon \cdot (g(p) \cdot p) - K_1 ,$$

$$\mu^*(|p|) \leq K_1 \cdot \mu_*(|p|) + K_2 ;$$

$$|g(p) \cdot p| \leq K_1 \cdot G(p) + K_2 \cdot (|p|^2 + 1), \quad \text{where again } \nabla_p G(p) = g(p) .$$

What kept us from including these assumptions were the technical problems that arise when one tries to approximate g by suitable functions g^M.

4. The Dirichlet Problem

In this section we want to show how to improve our results in the case of Dirichlet boundary data $u|_{\partial\Omega} \equiv$ const. by a modification of the method in Section 3. The principal tool is the

Lemma 4.1: Let $\Omega \subset \mathbb{R}^n$ be bounded, $\partial\Omega$ of class C^2. Let $g : [0,\infty) \to \mathbb{R}$ be locally Lipschitz continuous, and let $u \in C^2(\bar{\Omega},\mathbb{R})$, $u|_{\partial\Omega} \equiv 0$. Then

$$\int_\Omega \Delta_x u \cdot \operatorname{div}_x(g(|\nabla_x u|)\nabla_x u) = \sum_{i,j} \int_\Omega \partial_i\partial_j u \partial_j(g(|\nabla_x u|)\partial_i u) + \tag{4.1}$$

$$+ \int_{\partial\Omega} g(\partial_\nu u) \cdot |\partial_\nu u|^2 \cdot (n-1) \cdot H ,$$

where ν is the outward normal on $\partial\Omega$ and H the mean curvature with respect to ν.

Proof: An integration by parts gives

$$\int_\Omega \Delta_x u \cdot \operatorname{div}_x(g(|\nabla_x u|)\nabla_x u) = \sum_{i,j} \int_\Omega \partial_i\partial_j u \cdot \partial_j(g(|\nabla_x u|)\partial_i u) +$$

$$+ \int_{\partial\Omega} (\Delta_x u \cdot \partial_\nu u - \sum_{i,j} \partial_i\partial_j u \cdot \partial_j u \cdot \nu_i) \cdot g(|\nabla_x u|) .$$

Since $\nabla_x u = \partial_\nu u \cdot \nu$ on $\partial\Omega$, the boundary term can be written as

$$\int_{\partial\Omega} (\Delta_x u - \nu^T \cdot \nabla_x^2 u \cdot \nu) \cdot g(|\partial_\nu u|) \cdot \partial_\nu u .$$

Consider a point $x \in \partial\Omega$. After a suitable translation and rotation we can assume that $x = 0$ and locally about 0 $\partial\Omega = \{(\bar{x},x_n) | x_n = \phi(\bar{x})\}$, $\Omega = \{(\bar{x},x_n) | x_n < \phi(\bar{x})\}$, $\phi : U \to \mathbb{R}$ a C^2-function, U some neighborhood of $0 \in \mathbb{R}^{n-1}$, and $\nabla_{\bar{x}}\phi(0) = 0$. Then in these local coordinates

$$\Delta_x u(0) = \partial_n^2 u(0) - \Delta_{\bar{x}}\phi(0) \cdot \partial_n u(0) ,$$

$$\nu(0)^T \cdot \nabla^2 u(0) \cdot \nu(0) = \partial_n^2 u(0) .$$

But $\partial_n u(0) = \partial_\nu u(x)$ according to our choice of coordinate system, and $-\Delta_{\bar{x}}\phi(0) = (n-1) \cdot H(x)$, since $\nabla_{\bar{x}}\phi(0) = 0$. Hence

$$\int_{\partial\Omega} (\Delta_x u - \nu^T \cdot \nabla_x^2 u \cdot \nu) \cdot g(|\partial_\nu u|) \cdot \partial_\nu u = \int_{\partial\Omega} (n-1) \cdot H \cdot g(|\partial_\nu u|)|\partial_\nu u|^2 .$$

This Lemma is a step towards a simple non-linear version of Sobolevskii-type estimates for linear second-order elliptic operators as stated, e.g., in [3]. We thank Prof. A. Friedman for pointing this out to us.

Obviously the identity of (4.1) still holds under the assumptions $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, g locally Lipschitz and bounded, $r \mapsto g'(r) \cdot r$ bounded, by a standard approximation argument.

It is now possible to use (4.1) to modify the arguments of Section 3, if in (1.1) Γ is empty and $u|_{\partial\Omega\times[0,T]} \equiv 0$.

In step 2 of the proof of Theorem 3.1, the expression

$$\int_\Omega \Delta_x u^M \cdot \operatorname{div}(\tilde{g}(|\nabla_x u^M|)\nabla_x u^M) \cdot \xi^2$$

had to be estimated from below, $\xi \in W_0^{1,\infty}(\Omega)$ in order to take care of boundary terms. If $\partial\Omega$ is C^2-smooth and $u^M|_{\partial\Omega} \equiv 0$, we simply choose $\xi \equiv 1$ and get by Lemma 4.1 and manipulations similar to those in step 2

$$\int_\Omega \Delta_x u^M \cdot \operatorname{div}(\tilde{g}(|\nabla_x u^M|)\nabla_x u^M) \geq \kappa \cdot \int_\Omega \sum_{i,j} |\partial_i(\sqrt{\tilde{g}(|\nabla_x u^M|)}\ \partial_j u^M|^2$$

$$- C \cdot \int_\Omega \tilde{g}(|\nabla_x u^M|) \cdot |\nabla_x u^M|^2 + \int_{\partial\Omega} \tilde{g}(|\partial_\nu u^M|) \cdot |\partial_\nu u^M|^2 \cdot (n-1) \cdot H ,$$

$\kappa > 0$ a small constant. By a standard trace theorem this can be estimated from below by

$$\frac{\kappa}{2} \int_\Omega \sum_{i,j} |\partial_i(\sqrt{\tilde{g}(|\nabla u^M|)}\ \partial_j u^M)|^2 - C \cdot \int_\Omega \tilde{g}(|\nabla u^M|)|\nabla u^M|^2$$

(see Lemma 5.1), $C > 0$, and $\int_\Omega \tilde{g}(|\nabla u^M|) \cdot |\nabla u^M|^2$ is a priori bounded by (3.4) and Lemma 5.6. Hence in this special situation the solutions found in Theorem 3.1 fulfill

$$\int_0^T \int_\Omega |\nabla_x(\sqrt{g_0(|\nabla_x u|) + L}\ \nabla_x u)|^2 + \sup_{[0,T]} \int_\Omega |\nabla_x^2 u|^2(\cdot,t) < \infty . \tag{4.2}$$

However, one can also use Lemma 4.1 to show the existence of solutions of (I) for more general nonlinearities g_0, if the mean curvature of $\partial\Omega$ is non-negative. A possible class is described in the hypothesis

(B5) The function $g : \mathbb{R}^n \to \mathbb{R}^n$ is given by $g(\xi) = g_0(|\xi|) \cdot \xi$, $g_0 : [0,\infty) \to \mathbb{R}$ being locally Lipschitz-continuous on $(0,\infty)$. There exists a constant $L > 0$ such that

$$\frac{d}{dr}((g_0(r) + L) \cdot r) > 0 \quad \text{on} \quad (0,\infty) . \tag{4.3}$$

For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $r > 0$

$$|g_0(r) \cdot r| < \varepsilon \cdot \int_0^r (g_0(s) + L) \cdot s\,ds + C_\varepsilon . \tag{4.4}$$

We then get the

Theorem 4.2: Let $\Omega \subset \mathbb{R}^n$ be bounded, $\partial\Omega$ C^2-smooth, with non-negative mean curvature H (with respect to the outer normal). Let (B5), (A3), (A4) and (A5) hold with $u_0 \in W_0^{1,2}(\Omega)$ not depending on t. Then the equation (I) with boundary conditions

$$u|_{\partial\Omega\times[0,T]} \equiv 0 \tag{4.5}$$

has a distributional solution u satisfying (3.4) and

$$\sup_{[0,T]} \int_\Omega |\nabla_x^2 u|^2(\cdot,t) < \infty . \tag{4.6}$$

Sketch of the proof:

As in the proof of Theorem 3.1, we define for $M > 0$

$$g_0^M(r) = \inf(g_0(r),M),\ g^M(\xi) = g_0^M(|\xi|) \cdot \xi .$$

Then g_0^M still satifies (B5). We solve (I) with g replaced by g^M and get unique distributional solutions u^M that satisfy (3.14). By an argument similar to the one used above, $u^M \in L^\infty(0,T;W^{2,2}(\Omega))$ for all M. Still following the lines of the proof of Theorem 3.1, we apply the resolvent kernel of $\dot{a}$ to the equation (I) to get (3.9), differentiate formally (take difference quotients), and multiply with $-\Delta_x u^M$ which is an admissible test function. This gives the identity (3.10) and - after rearranging terms - (3.11) with $\xi \equiv 1$. We then apply Lemma 4.1 and conclude that

$$\int_\Omega \Delta_x u^M \mathrm{div}((g_0^M(|\nabla_x u^M) + L) \cdot \nabla_x u^M) > 0 ,$$

This Lemma is a step towards a simple non-linear version of Sobolevskii-type estimates for linear second-order elliptic operators as stated, e.g., in [3]. We thank Prof. A. Friedman for pointing this out to us.

Obviously the identity of (4.1) still holds under the assumptions $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, g locally Lipschitz and bounded, $r \mapsto g'(r) \cdot r$ bounded, by a standard approximation argument.

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(see Lemma 5.1), $C > 0$, and $\int_\Omega \tilde{g}(|\nabla u^M|) \cdot |\nabla u^M|^2$ is a priori bounded by (3.4) and Lemma 5.6. Hence in this special situation the solutions found in Theorem 3.1 fulfill

$$\int_0^T \int_\Omega |\nabla_x(\sqrt{g_0(|\nabla_x u|) + L}\, \nabla_x u)|^2 + \sup_{[0,T]} \int_\Omega |\nabla_x^2 u|^2(\cdot,t) < \infty . \tag{4.2}$$

However, one can also use Lemma 4.1 to show the existence of solutions of (I) for more general nonlinearities g_0, if the mean curvature of $\partial\Omega$ is non-negative. A possible class is described in the hypothesis

(<u>B5</u>) The function $g : \mathbb{R}^n \to \mathbb{R}^n$ is given by $g(\xi) = g_0(|\xi|) \cdot \xi$, $g_0 : [0,\infty) \to \mathbb{R}$ being locally Lipschitz-continuous on $(0,\infty)$. There exists a constant $L > 0$ such that

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<u>Theorem 4.2</u>: Let $\Omega \subset \mathbb{R}^n$ be bounded, $\partial\Omega$ C^2-smooth, with non-negative mean curvature H (with respect to the outer normal). Let (B5), (A3), (A4) and (A5) hold with $u_0 \in W_0^{1,2}(\Omega)$ not depending on t. Then the equation (I) with boundary conditions

$$u|_{\partial\Omega\times[0,T]} \equiv 0 \tag{4.5}$$

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<u>Sketch of the proof</u>:

As in the proof of Theorem 3.1, we define for $M > 0$

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Then g_0^M still satifies (<u>B5</u>). We solve (I) with g replaced by g^M and get unique distributional solutions u^M that satisfy (3.14). By an argument similar to the one used above, $u^M \in L^\infty(0,T;W^{2,2}(\Omega))$ for all M. Still following the lines of the proof of Theorem 3.1, we apply the resolvent kernel of $\dot{a}$ to the equation (I) to get (3.9), differentiate formally (take difference quotients), and multiply with $-\Delta_x u^M$ which is an admissible test function. This gives the identity (3.10) and - after rearranging terms - (3.11) with $\xi \equiv 1$. We then apply Lemma 4.1 and conclude that

$$\int_\Omega \Delta_x u^M \operatorname{div}((g_0^M(|\nabla_x u^M) + L) \cdot \nabla_x u^M) \ge 0 ,$$

using (4.3) and $H \geq 0$. Hence

$$\sup_{[0,T]} \int_\Omega |\Delta_x u^M|^2 \leq K < \infty \tag{4.7}$$

for all M, thus the u^M are uniformly bounded in $L^\infty(0,T;W^{2,2}(\Omega))$.

We now extract a subsequence of the $(u^M)_{M\geq 1}$ such that

(i) $u^M \to u$ strongly in $L^2(0,T;W^{1,2}(\Omega))$, $\nabla_x u^M \to \nabla_x u$ a.e. in $\Omega \times [0,T]$,

(ii) $g_0^M(|\nabla_x u^M|) \cdot \nabla_x u^M \to \xi$ weakly in $L^1(0,T;L^1(\Omega))$,

(iii) $\partial_t u^M \to \partial_t u$ weakly in $L^2(0,T;W^{1,2}(\Omega))$.

The choice (i) is possible, since the bounds (3.4) and (4.7) hold uniformly in M; properties of g^M (cf. Lemma 5.4) and the estimate (3.14) show that the $g_0^M(|\nabla u^M|)\nabla u^M$ are equi-integrable and hence weakly precompact in $L^1(0,T;L^1(\Omega))$, hence (ii) is possible. The continuity of g_0 then shows that

$$\xi = g_0(|\nabla_x u|) \cdot \nabla_x u \quad \text{a.e. on } \Omega \times [0,T] ,$$

and (4.7) still holds for the limit function. This proves Theorem 4.2.

<u>Remark</u>: The condition of "non-negative mean curvature of the boundary" that was used in Theorem 4.2 reminds of the general curvature conditions that guarantee classical solvability of quasilinear equations (cf. [22]). The "stationary" solutions of (I) are of this type, and it would be interesting to link these boundary conditions and properties of the kernel a to show the convergence of solutions as $t \to \infty$.

To conclude, we would like to comment on some related questions concerning the problem (I) or its variants.

<u>Existence of classical solutions</u>:

By means of contraction type arguments, one easily shows the existence of classical (C^2-) solutions for (I) and smooth data that exist locally in time, and these solutions will be unique. Our a priori estimates only permit to continue them in the case of one space dimension, however, since then $\partial_x u$ will be Hölder-continuous (by the estimates of Section 3 or 4), and one can apply the regularity theory for linear parabolic equations. Note that in one space dimension the introduction of the cut-off function ξ (in Section 3) is not necessary.

Uniqueness of solutions:

This will follow if one can show that the spatial gradients of solutions are a priori bounded on $\Omega \times [0,T]$ (and thus the unique approximating solutions u^M in Sections 3 and 4, obtained by modifying g and β for large arguments, become M-independent for large M). However, our estimates only guarantee (in the setting of Section 4) that

$$\nabla_x u \in L^\infty(0,T;L^\rho(\Omega))$$

with $\rho < \infty$ for $n = 2$ and $\rho = \frac{2n}{n-2}$ for $n > 2$. The usual "bootstrapping" techniques (which would amount to regarding the integral term as a perturbation of a linear equation) will not work due to the high order (of growth and differentiation) of the integral term.

The case of $g(u,\nabla_x u)$:

If the integral term $a * \operatorname{div}_x g(\nabla_x u)$ is replaced by, e.g., $a * \operatorname{div}_x(g(u) \cdot \nabla_x u)$, then existence arguments become in fact simpler; since for approximating solutions u^M one only has to guarantee the strong convergence of, e.g., u^M in some L^p, but not of $\nabla_x u^M$. Hence a priori estimates of $\nabla_x u^M$ are sufficient to do this; they can be obtained (under suitable additional assumptions) by taking $g(u^M)$ as a test function in (I) and using some definiteness properties of the form

$$v \to \int_0^R a * v(t) \cdot v(t)dt \quad \text{for} \quad v \in L^2(0,T;\mathbb{R}^n) .$$

The more general case $a * \operatorname{div}_x(g(u,\nabla_x u))$ seems to be more difficult.

Appendix

Here we state some auxiliary arguments that have been used in the previous proofs.

Lemma 5.1: Let $\Omega \subset \mathbb{R}^n$ be bounded, $\partial\Omega$ of class $C^{0,1}$, $\Gamma \subset \partial\Omega$ an $(n-1)$-dimensional submanifold, Ω locally on one side of $\partial\Omega$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for all $u \in W^{1,2}(\Omega)$

(5.1) $$\int_\Gamma |u|^2 \leq \varepsilon \int_\Omega |\nabla_x u|^2 + C(\varepsilon) \int_\Omega |u|^2 .$$

This is a simple consequence of well-known trace theorems (cf. [16]).

Lemma 5.2: Let $\beta : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz-continuous, let $L > 0$ and

$$-L + 1 \leq \beta'(r) \quad \text{for all} \quad r \in \mathbb{R} ,$$

and assume that for any $\varepsilon > 0$ there exists some $C(\varepsilon) > 0$ such that for all $r \in \mathbb{R}$

$$\beta'(r) \leq \varepsilon|\beta(r) + L \cdot r| + C(\varepsilon) .$$

Let $B(r) = \int_0^r \sqrt{\beta'(s) + L}\, ds$.

Then for any $\delta > 0$ there exists some $\tilde{C}(\delta) > 0$ such that for all $r \in \mathbb{R}$

$$|\beta(r)| \leq \delta \cdot B^2(r) + \tilde{C}(\delta) .$$

Proof: Let $r \in \mathbb{R}$, then

$$\begin{aligned} |\beta(r) + L \cdot r| &\leq |\beta(0)| + \int_0^r (\beta'(s) + L)ds \\ &\leq |\beta(0)| + \int_0^r \sqrt{\beta'(s) + L} \cdot \sqrt{\varepsilon|\beta(s) + L \cdot s| + C(\varepsilon)}\, ds \\ &\leq |\beta(0)| + B(r) \cdot \sqrt{\varepsilon|\beta(r) + Lr| + C(\varepsilon)} \\ &\leq |\beta(0)| + \varepsilon B^2(r) + \frac{C(\varepsilon)}{4\varepsilon} + \frac{1}{4}|\beta(r) + Lr| , \end{aligned}$$

hence

$$\begin{aligned} |\beta(r)| &\leq \varepsilon \cdot B^2(r) + C_1(\varepsilon) + L \cdot |r| \\ &\leq 2\varepsilon B^2(r) + \bar{C}(\varepsilon) . \end{aligned}$$

__Lemma 5.3:__ Let $G : \mathbb{R}^n \to \mathbb{R}$ be differentiable, let $L > 0$, let $\tilde{G}(\xi) = G(\xi) + \frac{L}{2}(|\xi|^2 + 1)$ be positive, and let $g = \nabla_\xi G$ satisfy any of the following hypotheses:

(i) There exists $C > 0$ such that for all ξ, η

$$|g(\xi) \cdot \eta| \le C \cdot (\tilde{G}(\xi) + \tilde{G}(\eta) + 1) .$$

(ii) The function g is given by $g(\xi) = g_0(|\xi|) \cdot \xi$, g_0 is locally Lipschitz continuous, and there exists a constant $C > 0$ such that for all $r > 0$

$$g_0(r) + L > 0 ,$$

$$0 \le \frac{d}{dr}((g_0(r) + L) \cdot r) \le C \cdot (g_0(r) + L) .$$

Then for all $\delta > 0$ there exists $\tilde{C}(\delta) > 0$ such that for all $\xi \in \mathbb{R}^n$

$$|g(\xi)| \le \delta \cdot \tilde{G}(\xi) + \tilde{C}(\delta) .$$

Condition (ii) implies condition (i).

__Proof:__ In case (i), let $\bar{C}(\delta) = \max\limits_{\delta \cdot |\eta| \le 1} \tilde{G}(\eta)$. Then for any ξ

$$|g(\xi)| = \delta \cdot \max_{\delta \cdot |\eta| \le 1} |g(\xi) \cdot \eta| \le \delta \cdot C_1 \cdot \tilde{G}(\xi) + \delta \cdot C_1 \cdot (\bar{C}(\delta) + 1) .$$

In case (ii), put $\bar{g}(r) = g_0(r) + L$. Then for $\eta, \xi \in \mathbb{R}^n$ and $r_0 = \max\{|\eta|, |\xi|\}$

$$|g(\xi) \cdot \eta| \le \bar{g}(|\xi|) \cdot |\xi| \cdot |\eta| + L \cdot |\xi| \cdot |\eta|$$

$$\le \bar{g}(r_0) \cdot r_0^2 + L \cdot |\xi| \cdot |\eta|$$

$$= \int_0^{r_0} (\bar{g}(s) \cdot s)' \cdot s\,ds + \int_0^{r_0} \bar{g}(s) \cdot s\,ds + L \cdot |\xi| \cdot |\eta|$$

$$\le (C_2 + 1) \int_0^{r_0} \bar{g}(s) \cdot s\,ds + L \cdot |\xi||\eta|$$

$$\le C + (C_2 + 1) \cdot \int_0^{r_0} \bar{g}(s) \cdot s\,ds + \frac{L}{2}(|\xi|^2 + |\eta|^2)$$

$$\le C \cdot (\tilde{G}(\xi) + \tilde{G}(\eta) + 1)$$

and we are back in case (i).

Lemma 5.4: Let Ω be a finite measure space, $M \subset L^1(\Omega,\mathbb{R})$ a bounded set and $N \subset L^1(\Omega,\mathbb{R}^n)$. For all $u \in N$ and $\varepsilon > 0$ let there exist $v \in M$ and $C_\varepsilon > 0$ such that for a.e. $x \in \Omega$

$$|u(x)| \leq \varepsilon \cdot |v(x)| + C_\varepsilon .$$

Then N is equi-integrable (and hence weakly sequentially compact in $L^1(\Omega,\mathbb{R}^n)$), i.e.

$$\lim_{k\to\infty} \int_{\{|u|\geq k\}} |u| = 0 \quad \text{uniformly in } u \in N .$$

Proof:

$$\int_{\{|u|\geq k\}} |u| \leq \int_{\{|u|\geq k\}} (\varepsilon|v| + C_\varepsilon) \leq \varepsilon \cdot M + \frac{C_\varepsilon \cdot M_1}{k} ,$$

where

$$M = \sup_{v\in M} \int_\Omega |v|, \quad M_1 = \sup_{u\in N} \int_\Omega |u| \leq M + C_1 \cdot \int_\Omega 1 .$$

Lemma 5.5: Let $(w^m)_{m\geq 1}$ be a sequence in $L^\infty([0,T] \times \Omega,\mathbb{R}^n)$, such that (in the notation of Section 2)

$$g(w^m) \to \zeta \quad \text{weakly in } L^1([0,T] \times \Omega,\mathbb{R}^n)$$

and

$$\operatorname*{ess\,sup}_{[0,T]} \int_\Omega (G(w^m) + L \cdot |w^m|^2)(\cdot,t) \leq K < \infty$$

for all m.

Let $v \in L^1([0,T] \times \Omega,\mathbb{R}^n)$, $\operatorname*{ess\,sup}_{[0,T]} \int_\Omega (G(v) + L|v|^2)(\cdot,t) < \infty$. Then

$$\int_\Omega g(w^m) \cdot v(\cdot) \to \int_\Omega \zeta \cdot v(\cdot) \quad \text{weakly in } L^1(0,T;\mathbb{R}) .$$

Proof: For $M > 0$, define $v_M = \inf\{M, \sup\{-M, v\}\}$. Let $\phi \in L^\infty(0,T;\mathbb{R})$, then

$$(5.2) \qquad \int_0^T \phi(t) \cdot \int_\Omega (g(w^m) \cdot v)(\cdot,t)dt = \int_0^T \phi(t) \cdot \int_\Omega g(w^m) \cdot (v - v_M)(\cdot,t)dt$$

$$+ \int_0^T \phi(t) \cdot \int_\Omega g(w^m) \cdot (v_M)(\cdot,t)dt .$$

The first term can be estimated by

$$(5.3) \qquad \frac{1}{R} \cdot \|\phi\|_{L^\infty} \cdot C \cdot \int_0^T\int_\Omega (\tilde{G}(w^m) + \tilde{G}(R(v - v_M)) + 1)$$

by (2.1), $\tilde{G}(p) = G(p) + \frac{L}{2}|p|^2 + 1$, for any $R > 1$, since (2.1) also implies

$$\tilde{G}(K \cdot p) \le (C + \tilde{G}(p)) \cdot e^{K \cdot C} .$$

Hence the expression (5.3) is bounded by

$$\frac{1}{R} \cdot C \cdot \int_0^T\int_\Omega (\tilde{G}(R \cdot (v - v_M)) + 1) .$$

Since $\int_0^T\int_\Omega \tilde{G}(R \cdot (v - v^M)) \to T \cdot \int_\Omega \tilde{G}(0)$, as $M \to \infty$, for any R, we thus derive from (5.2) and the convergence of the second term in (5.2)

$$\limsup_{m\to\infty} \left|\int_0^T \phi(t) \cdot \int_\Omega (g(w^m) \cdot v - \zeta \cdot v_M)dt\right| \le \frac{1}{R} \cdot C$$

for any R and M, which shows the claim.

Lemma 5.6: Let $g_0 : [0,\infty) \to \mathbb{R}$ be as in (B2), $G_0(r) = \int_0^r g_0(s) \cdot s ds$, L_1 as in (B2). Then for all $\varepsilon > 0$ there exists some $C(\varepsilon) > 0$ such that for all $r,s > 0$

$$|g_0(r) \cdot r \cdot s| \le \varepsilon \cdot (G_0(s) + L_1 s^2) + C(\varepsilon)(G_0(r) + L_1 \cdot r^2) .$$

Proof: First note that $r \mapsto (g_0(r) + L_1) \cdot r$ is monotone and (without loss of generality) positive for $r > 0$. Further for $r_2 > r_1 > 0$

$$\frac{(g_0(r_2) + L_1) \cdot r_2}{(g_0(r_1) + L_1) \cdot r_1} = \exp\Big(\int_{r_1}^{r_2} \frac{g_0^1(\tau) \cdot \tau + g_0(\tau) + L_1}{(g_0(\tau) + L_1) \cdot \tau} \, d\tau\Big) <$$

$$< \exp\Big(\int_{r_1}^{r_2} \frac{C + 1}{\tau} \, d\tau\Big) = \Big(\frac{r_2}{r_1}\Big)^{C+1} .$$

So either $\varepsilon \cdot r > s$; then

$$(g_0(r) + L_1) \cdot r \cdot s < \varepsilon \cdot (g_0(r) + L_1) \cdot r^2 ,$$

or $r < \frac{s}{\varepsilon}$; then

$$(g_0(r) + L_1) \cdot r \cdot s < \Big(\frac{1}{\varepsilon}\Big)^{C+1} \cdot (g_0(s) + L_1) \cdot s^2 .$$

On the other hand,

$$G_0(r) + \frac{L_1}{2} \cdot r^2 = \int_0^r (g_0(s) + L_1) \cdot s\,ds = \frac{1}{2} (g_0(r) + L_1) \cdot r^2 - \frac{1}{2} \int_0^r g_0'(s) \cdot s^2 ds$$

$$> \frac{1}{2} (g_0(r) + L_1) \cdot r^2 - \frac{C}{2} \int_0^r (g_0(s) + L_1) \cdot s\,ds$$

$$= \frac{1}{2} (g_0(r) + L_1) \cdot r^2 - \frac{C}{2} \cdot \Big(G_0(r) + \frac{L_1}{2} r^2\Big) ,$$

hence

$$(g_0(r) + L_1) \cdot r^2 < (C + 2) \cdot (G_0(r) + \frac{L_1}{2} r^2) .$$

Hence

$$0 < (g_0(r) + L_1) \cdot r \cdot s < (C + 2) \cdot \varepsilon \cdot (G_0(r) + \frac{L_1}{2} r^2) + \frac{C + 2}{\varepsilon^{C+1}} (G_0(s) + \frac{L_1}{2} s^2) .$$

Combining this with

$$|r \cdot s| < \delta \cdot r^2 + \frac{1}{4\delta} s^2 \quad \text{for all} \quad \delta$$

gives the desired estimate.

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We study partial integro-differential equations of the type

$$\text{(I)} \qquad \partial_t u(\cdot,t) + Au(\cdot,t) + \int_0^t a(t-s)Bu(\cdot,s)ds = f(\cdot,t), \quad 0 \le t \le T ,$$

in some spatial domain $\Omega \subset \mathbb{R}^n$, A being a linear and B a quasilinear elliptic operator of second order, both in divergence form, together with initial and various boundary conditions. We give conditions on the structure

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20. ABSTRACT - cont'd.

of A and B that lead to a priori estimates and show how to get the existence of weak solutions ($u(\cdot,t) \in W^{1,p}(\Omega)$ or $u(\cdot,t) \in W^{2,2}_{loc}(\Omega)$ for a.e. t) from approximating solutions (that solve finite-dimensional versions of (I) or versions with modified coefficients). The main tools are "energy" estimates on $\|\partial_t u(\cdot,t)\|^2_{L^2} + \int_\Omega G(\nabla_x u)$, if $Bu = -\mathrm{div}_x(\nabla_u G(\nabla_x u))$, for $W^{1,p}$-solutions, and estimates on the L^2-product $\langle Au,Bu\rangle_{L^2}$ for $W^{2,2}_{loc}$-solutions.